Physics 523, General Relativity

Homework 3

Due Monday, 16th October 2006

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Problem 1

a) Let us consider the region of the $t-x$-plane which is bounded by the lines $t=0, t=1, x=0$, and $x=1$; we are to find the unit outward normal one-forms and their associated vectors for each of the boundary lines.

It is not hard to see that the unit outward normal one-forms and their associated vectors are given by

$$t=0: \quad -dt \mapsto \vec{e}_t \qquad\qquad t=1: \quad dt \mapsto -\vec{e}_t; \tag{1.a.1}$$

$$x=0: \quad -dx \mapsto -\vec{e}_x \qquad\qquad x=1: \quad dx \mapsto \vec{e}_x. \tag{1.a.2}$$

b) Let us now consider the triangular region bounded by events with coordinates $(1,0),(1,1)$, and $(2,1)$; we are to find the outward normal for the null boundary and its associated vector.

The equation for the null boundary of the region is $t=x+1$, which is specified by the vanishing of the function $t-x-1=0$. The normal to the surface is simply the gradient of this zero-form, and so the normal is

$$dt - dx, \tag{1.b.3}$$

and the associated vector is

$$-\vec{e}_t - \vec{e}_x. \tag{1.b.4}$$

Problem 2

We are to describe the (proper orthochronous) Lorentz-invariant quantities that can be built out of the electromagnetic field strength F_{ab} and express these invariant in terms of the electric and magnetic fields.

Basically, any full contraction of indices will result in a Lorentz-invariant quantity. Furthermore, because we are considering things which are invariant under only proper orthochronous transformation, we are free to consider CP-odd combinations, which mix up components with their Hodge-duals. A list of such invariants are:

$$F^a{}_a = 0 \qquad F^{ab}F_{ab} = -\left(\star F^{ab}\right)\left(\star F_{ab}\right) = 2\left(\vec{B}^2 - \vec{E}^2\right) \qquad \left(\star F^{ab}\right)F_{ab} = -4\vec{E}\cdot\vec{B}. \tag{2.a.1}$$

This does not exhaust the list of invariants, however: we are also free to take a number of derivatives. These start getting rather horrendous, but we can start with an easy example:

$$\left(\partial_a F^{ab}\right)^2 = 16\pi^2 J^a J_a = 16\pi^2\left(\vec{J}^2 - \rho^2\right). \tag{2.a.2}$$

Along this vein, we find

$$\left(\partial_a F^{ab}\right)\left(\partial_c F^{cd}\right)F_{bd} = 16\pi^2 \vec{J}\cdot\left(\vec{B}\times\vec{J}\right); \tag{2.a.3}$$

$$\left[\left(\partial_a F^{ab}\right)F^{ab}\right]^2 = 16\pi^2\left\{\rho^2\vec{E}^2 - \left(\vec{E}\cdot\vec{J}\right)^2 + \left(\vec{B}\times\vec{J}\right)^2 - 2\rho\vec{E}\cdot\left(\vec{B}\times\vec{J}\right)\right\}; \tag{2.a.4}$$

$$\left(\partial_a F^{ab}\right)\left(\partial_c F^{cd}\right)\left(\star F_{bd}\right) = 16\pi^2\vec{J}\cdot\left(\vec{E}\times\vec{J}\right). \tag{2.a.5}$$

We could go higher in derivatives, but we know that $\partial_a \star F^{ab} = 0$ and we are free to make the Lorentz gauge choice $\Box F^{ab} = 0$. I suspect that further combinations will not yield independent quantities.

Problem 3

Consider a pair of twins are born somewhere in spacetime. One of the twins decides to explore the universe; she leaves her twin brother behind and begins to travel in the x-direction with constant acceleration $a = 10$ m/s^2 as measured in her rocket frame. After ten years according to her watch, she reverses the thrusters and begins to accelerate with a constant $-a$ for a while.

a) At what time on her watch should she again reverse her thrusters so she ends up home at rest?

There is an obvious symmetry in this problem: if it took her 10 years by her watch to go from rest to her present state, then 10 years of reverse acceleration will bring her to rest, at her farthest point from home. Because of the constant negative acceleration, after reaching her destination at 20 years, she will begin to accelerate towards home again. In 10 more years, when her watch reads 30 years, she will be in the same state as when her watch read 10 years, only going in the opposite direction.

Therefore, at 30 years, she should reverse her thrusters again so she arrives home in her home's rest frame.

b) According to her twin brother left behind, what was the most distant point on her trip?

To do this, we need only to solve the equations for the travelling twin's position and time as seen in the stationary twin's frame. This was largely done in class but, in brief, we know that her four-acceleration is normal to her velocity: $a^\xi u_\xi = 0$ everywhere along her trip, and $a^\xi a_\xi = a^2$ is constant. This leads us to conclude that

$$a^t = \frac{du^t}{d\tau} = au^x \qquad \text{and} \qquad a^x = \frac{du^x}{d\tau} = au^t, \tag{3.b.1}$$

where τ is the proper time as observed by the travelling twin. This system is quickly solved for an appropriate choice of origin[1]:

$$t = \frac{1}{a}\sinh(a\tau) \qquad \text{and} \qquad x = \frac{1}{a}\cosh(a\tau). \tag{3.b.2}$$

This is valid for the first quarter of the twin's trip—all four 'legs' can be given explicitly by gluing together segments built out of the above.

For the purposes of calculating, it is necessary to make $a\tau$ dimensionless. This is done simply by

$$a = \frac{10\text{ m}}{\text{sec}^2} = 1.053\text{ year}^{-1}. \tag{3.b.3}$$

An approximate result[2] could have been obtained by thinking of $c = 3\times 10^8$ m/s and 3×10^7 sec $= 1$ year.

So the distance at 10 years is simply

$$x(10\text{ yr}) = \frac{1}{1.053}\cosh(10.53) = 17710\text{ light years.}$$

The maximum distance travelled by the twin as observed by her (long-deceased) brother is therefore twice this distance, or[3]

$$\max(x) = 35,420\text{ light years.} \tag{3.b.4}$$

c) When the sister returns, who is older, and by how much?

Well, in the brother's rest frame, his sister's trip took four legs, each requiring

$$t(10\text{ yr}) = \frac{1}{1.053}\sinh(10.53) = 17710\text{ years,}$$

which means that

$$t_{\text{total}} = 70,838\text{ years.} \tag{3.c.5}$$

In contrast, his sister's time was simply her proper time, or 40 years. Therefore the brother who stayed behind is now 70,798 years older than his twin sister.

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[1] We consider the twin to begin at $(t = 0, x = 1)$.

[2] Because cosh goes like an exponential for large argument, our result is exponentially sensitive to the figures; because we know c and the number of seconds per year to rather high-precision, there is no reason not to use the correct value of a—indeed, the approximate value of $a \sim 1$ year^{-1} gives an answer almost 40% below our answer.

[3] If we had used instead $a = 1$/year as encouraged by the problem set, our answer would have been 22,027 light years.

Problem 4[4]

Consider a star located at the origin in its rest frame $\mathscr{O}$ emitting a continuous flux of radiation, specified by luminosity L.

a) We are to determine the non-vanishing components of the stress-energy tensor as seen by an observer located a distance x from the star along the x-axis of the star's frame.

There are many ways to go about determining the components of the stress-energy tensor. We will be un-inspired and compute it directly from the equation for the Maxwell stress-energy tensor (found by looking at metric variations of the Maxwell action):

$$T^{ab} = F^{a}_{\ c}F^{bc} - \frac{1}{4}\eta^{ab}F_{ab}F^{ab}. \tag{4.a.1}$$

We have in previous exercises computed all of the necessary terms, so we may simply quote that

$$T^{00} = \vec{E}^2 + \frac{1}{2}\left(\vec{B}^2 - \vec{E}^2\right) = \frac{1}{2}\left(\vec{B}^2 + \vec{E}^2\right) = |\vec{S}|, \tag{4.a.2}$$

$$T^{0i} = \left(\vec{E}\times\vec{B}\right)^i = |\vec{S}|, \tag{4.a.3}$$

where $\vec{S}$ is the Poynting vector, whose magnitude is just the energy density flux. Now, when we expand T^{xx}, we find a bit more work in for us, at first glance, we see

$$T^{xx} = \frac{3}{2}E_x^2 - \frac{1}{2}B_x^2 + \frac{1}{2}\left(E_y^2 + E_z^2 + B_y^2 + B_z^2\right);$$

but we should note that because the radiation is only reaching the observer along the x-direction, $\vec{S}$ lies along the x-direction and so $B_x = E_x = 0$; therefore, we do indeed see that

$$T^{xx} = \frac{1}{2}\left(\vec{B}^2 + \vec{E}^2\right) = |\vec{S}|. \tag{4.a.4}$$

And making use of the fact that $\vec{S}$ only has components in the x-direction, we see that $T^{0y} = T^{0z} = 0$—with symmetrization implied.

Now, the energy density flux over a sphere centred about the origin of radius x naturally is $\frac{L}{4\pi x^2}$. Therefore, we see that

$$\therefore T^{00} = T^{x0} = T^{0x} = T^{xx} = \frac{L}{4\pi x^2}. \tag{4.a.5}$$

b) Let $\vec{X}$ be the null vector connecting the origin in $\mathscr{O}$ to event at which the radiation is measured. Let $\vec{U}$ be the velocity four-vector of the sun. We are to show that $\vec{X} \to (x, x, 0, 0)$ and that T^{ab} has the form

$$\mathbf{T} = \frac{L}{4\pi}\frac{\vec{X}\otimes\vec{X}}{\left(\vec{U}\cdot\vec{X}\right)^4}.$$

Well, it is intuitively obvious that if an observer sees radiation at $(x, x, 0, 0)$, that, because it is null and forward-propagating, it must have been emitted from a source along the line $\tau(1, 1, 0, 0)$ where τ is an affine parameter for the world line of the photon. If it is the case that the photon was emitted by the sun that is sitting at $x = 0$, then it must have been emitted at $(0, 0, 0, 0)$, which means that $\vec{X} \to (x, x, 0, 0)$.

Now, using the fact that $\vec{U} = (1, 0, 0, 0)$ for the star, we have that $\vec{U}\cdot\vec{X} = x$, and this is frame-independent. Now, we see that $\vec{X}\otimes\vec{X}$ only has components in (t, x)-directions

[4] This is the most poorly worded problem I have encountered thus far in this course. If there is any misunderstanding, I am strongly inclined to blame Schutz.

and furthermore all the coefficients are the same, namely x^2. Therefore

$$\frac{L}{4\pi}\frac{\vec{X}\otimes\vec{X}}{\left(\vec{U}\cdot\vec{X}\right)^4}=\frac{L}{4\pi x^2}\left(\vec{e}_t\otimes\vec{e}_t+\vec{e}_t\otimes\vec{e}_t+\vec{e}_x\otimes\vec{e}_t+\vec{e}_x\otimes\vec{e}_x\right). \tag{4.b.6}$$

Because this matches our explicit calculation in a certain frame and the expression is manifestly frame-independent we see that this is a valid expression for $\mathbf{T}$ in any reference frame[5].

c) Consider an observer $\overline{\mathscr{O}}$ travelling with speed v away from the star's frame $\mathscr{O}$ in the x-direction. In that frame, the observation of radiation is at $\vec{X}\to(R,R,0,0)$. We are to find R as a function of x and express $T^{\bar{0}\bar{x}}$ in terms of R.

There is no need to convert $\vec{U}$ of the sun into $\overline{\mathscr{O}}$'s coordinates because it only appears in $\mathbf{T}$ as a complete contraction—which is to say that $\vec{U}\cdot\vec{X}$ is frame independent. Now, all we need to do then is compute the coordinates of $\vec{X}$ in $\overline{\mathscr{O}}$'s coordinate system. This is done by a simple Lorentz transformation:

$$\vec{X}\underset{\overline{\mathscr{O}}}{\to}(x\gamma(1-v),x\gamma(1-v),0,0)\equiv(R,R,0,0), \tag{4.c.7}$$

which is to say, $R=x\gamma(1-v)$.

Bearing in mind that the numerator in the expression of $\mathbf{T}$ was invariant, we see that

$$T^{\bar{0}\bar{x}}=\frac{L}{4\pi}\frac{R^2}{x^4}. \tag{4.c.8}$$

Now, inverting our expression for R, we see that

$$x^2=\frac{R^2}{\gamma^2(1-v)^2}=R^2\left(\frac{1+v}{1-v}\right),$$

and so

$$\therefore T^{\bar{0}\bar{x}}=\frac{L}{4\pi R^2}\left(\frac{1-v}{1+v}\right)^2. \tag{4.c.9}$$

[5] Well, specifically, the difference between the T^{ab} calculated above and the coordinate-free tensor vanishes identically at x; this tensor identity is obviously frame independent and so the tensors are identical.